Developing Algebraic Thinking

By Suzanne Alejandre

How many times have you found activities, whether introduced in your textbook or presented at a conference or an in-service workshop, that seem to deal with patterning and sequential thinking but in which the formalized mathematics behind the activity is missing from the materials?

I have encountered elementary and middle school materials that contain engaging activities described as encouraging students' cooperative learning. These activities are often suggested for use at the beginning of the school year or as new groups are formed in the classroom, when in fact they are excellent opportunities for teaching algebraic thinking.

The **Traffic Jam** activity was one of those activities for me. I first encountered it as an icebreaker during the 1996 Math Forum Summer Institute. As the institute began and after a few brief introductions to the subsequent week's activities, the participants went out onto a porch, where the floor conveniently consisted of a grid of square tiles. The institute participants and staff divided into groups of six each to experiment with the activity.

I remember the problem as being one that I did not understand but that was a fun way to get to know people. After the institute, when I opened the textbook to start preparing for the opening of the school year, I was surprised to find the same activity in the book.

The problem involves seven stepping stones and six people. On the three left-hand stones, facing the center, stand three of the people. The other three people stand on the three right-hand stones, also facing the center. The center stone is not occupied. Everyone must move so that the people originally standing on the right-hand stepping stones are on the left-hand stones, and so that those originally standing on the left-hand stepping stones are on the right-hand stones, with the center stone again unoccupied.

Interestingly, the textbook version of the problem was also presented as an icebreaker activity. No expectation was conveyed that the students would learn any mathematics behind the activity.

The first year that I used Traffic Jam, I introduced it as I had first learned it and as it was written in the textbook – a full-body, kinesthetic activity with six students in each group. The result was chaos. The students had no idea how to cooperate or get a handle on the problem, let alone solve it mathematically. Mike Morton, a Swarthmore student also at the 1996 Summer Institute, had written a **Traffic Jam java applet**. I tried that version with the students, but in so doing, I still wasn't asking students to think of the mathematics behind the activity.

The following school year, however, as I reflected on appropriate uses of technology and how I could combine those uses with activities and get to a point of formalizing the mathematics, the **Traffic Jam Activity** started unfolding as a perfect example of a mathematically rich experience.

I found that combining the use of a manipulative activity with a technological activity both extended the time spent on the problem and gave students additional opportunities for understanding. Classroom discussion throughout the process was important, as were formalizing the mathematics and encouraging a connection between the real-world experience and the symbolic representation.

In the following pages I have tried to capture the experience that resulted in my students' getting to a point of full understanding of the kinesthetic activity and the algebra behind it.

Setting the Stage

Lesson:
Traffic Jam Activity Teacher Lesson Plan and Student Activities

Class:
31 seventh-grade students.

Classroom:
Four days a week in a computer lab with 20 Macintosh LC 580's, each with Internet access. One day a week in a classroom with tables and chairs arranged for groups of four students.

School:
Frisbie Middle School, with schoolwide Title I funding, in Rialto, California, which is about 70 miles east of Los Angeles.

When:
1999 – 2000 school year.

Videotape:
During the 6 days that we worked on this activity, a videotape was made to record my teaching and the students' interactions. The quotes that follow were transcribed from that videotape.

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Reading Aloud

I introduced the activity by passing out textbooks and small baskets containing plastic people figures. I explained that we would be working through this activity in several ways, including these:

Using a diagram in the textbook and plastic-people manipulatives

Using a Java applet from the Math Forum

As a paper-and-pencil activity

As a full-body activity, which would be videotaped

Some students volunteered to read from the textbook. Other students were called on to restate the information to check for understanding.

I used the overhead projector to demonstrate how to place the plastic people on the diagram in the textbook so that we could begin the activity.

Teacher: *There is actually an algebraic formula that we can learn so that if I give you any number of people (if I say you have five people on this team and five people on that team), you will automatically be able to tell me the minimum number of moves it takes to have all of that happen.*

Each group of four students was instructed to share the plastic people so that each pair of students had three of one color and three of another color.

Activities

Teacher: *Your task right now is to try this using the rules.*

Student: *So it's kind of like checkers?*

Teacher: *Yes, I guess you could say that.*

The students worked with partners to try the activity using the book and plastic people. One student moved the plastic people while the other student counted the moves being made.

The teacher's role was that of a facilitator. Some students hadn't quite understood that the activity was not a competitive game. The rules were reinforced, and other partners of students engaged in the activity were pointed out as models to follow.

The students tried hard to complete the activity, but no partners could give the correct answer by the end of the first class period.

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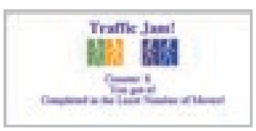
Applet

Some students worked together and others worked individually, using the Traffic Jam Java applet written by Mike Morton.



> **One student:** *"We got it on easy!"*

> **Another student:** *"The two is easy."*

The students were referring to the option of setting the applet so that two pairs of people are involved. Another option is to set the applet on **hard** so that four pairs of people are involved.

The students quickly noticed that when they correctly completed the activity, the computer provided feedback. Because some students were able to correctly complete the **easy** level, other students who had not moved from **medium** to **easy** quickly made the switch so that they could receive the positive feedback of success.

> **Student:** *It will still say it, though [referring to the computer's feedback].*

The students had the book and the plastic people available while they worked with the applet on the computer. The students spontaneously went back to the plastic people to try to complete the activity.

Here is how one student described what he was doing while moving his plastic people:

> "The little blue one to the right.
>
> Then you move the little yellow one to the left.
>
> Then you move the little yellow one to the left.
>
> Then move the little blue one to the right.
>
> Then you move the other little blue one to the right.
>
> Then you move the other little blue one to the right.
>
> Then you move the little yellow one to the left.
>
> Then you move the little yellow one to the left.
>
> Then the little yellow one.
>
> Move the little blue one to the right.
>
> Move the little blue one to the right.
>
> Move the other little blue one to the right.
>
> Move the little yellow one to the left.
>
> Move the other little yellow one to the left.
>
> Move the other little blue one to the right!"

After working with the applet, many pairs of students were able to successfully go back to the manipulatives and demonstrate that 15 moves are required to have the three teams change sides.

When asked how they had figured out the minimum number of moves for three pairs of people, one group•s explanation was this:

> "The way we figured it out is that we did all the spaces by numbers, and so this is 1, 2, 3, 4, 5, 6, 7. So then all the division worked out with what number space worked out, and so we had a pattern.
>
> We moved space 3 to 4 and then
>
> 5 to 3,
>
> 6 to 5,
>
> 4 to 6,
>
> 2 to 4,
>
> 1 to 2,
>
> 3 to 1,
>
> 5 to 3,
>
> 7 to 5,
>
> 6 to 7,
>
> 4 to 6,
>
> 2 to 4,
>
> 3 to 2,
>
> 5 to 3,
>
> 4 to 5,
>
> which is 15 moves altogether."

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Formalizing

On the third day of the lesson, after reading the problem, understanding the rules, working with manipulatives, and using the Java applet, the students were asked to make a chart in their math journals. The chart looked like this:

Number of Pairs	Number of People	Minimum Number of Moves
3	6	15

Number of Pairs	Number of People	Minimum Number of Moves
1	2	
2	4	
3	6	15
4	8	
5	10	
6	12	
7	14	
8	16	

The students followed the model on the whiteboard and filled in 3, 6, and 15.

After the header of the first column was pointed out, the students responded that the column could be filled to include 1, 2, 3, 4, 5, 6, 7, 8, and so on.

The students' attention was directed to the header of the middle column. One student said, "2, 4," and another student followed up with "6, 8." The teacher commented, "We're thinking in patterns already."

Now for the interesting part of the chart. The class' attention was focused on filling in the numbers in the third column.

> **Teacher:** *What do you think the answer is there [pointing to the minimum number of moves for one pair or two people].*
>
> **One student:** *Four.*
>
> **Teacher:** *Are you sure?*
>
> **Another student:** *Five.*
>
> **Teacher:** *How can we find out? Do you have two people, one pair?*

The students tried the task with their plastic people. One student answered, "Three," and justified the answer. A similar procedure was followed for the second row. All students had filled in the first three rows of the chart.

> **Teacher:** *Now I have a question: If I have four pairs, how many people would that be?*
>
> **A student:** *Eight people*
>
> **Teacher:** *The hard question is, how many moves?*
>
> **A student:** *Twenty.*
>
> **Teacher:** *Why did you just decide it was 20? If you are looking at the pattern, tell me what pattern you see.*
>
> **A student:** *The number of pairs is a pattern.*
>
> **Another student:** *You're adding the number again.*

The students' attention was brought back to the first and second columns to see whether the students could articulate the patterns.

> **Teacher:** *Do you see any pattern in the middle column? With the number of people?*
>
> **A student:** *2 + 2 is 4, 4 + 4 is 8, 8 + 8 is 16. Oh, 2, 4, 6, 8, 10, ...*

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Teacher: *In the first column, we just add "1" each time. What's happening here [pointing to the middle column]?*

A student: *Doubling.*

Another student: *Adding two.*

Teacher: *What's happening here [pointing to the third column]?*

Student 1: *You add the numbers, like 1 + 1 = 2.*

Student 2: *Adding 7.*

Student 3: *3 and 5 is 8, 8 plus 7 is 15.*

(Other students were thinking aloud.)

Teacher: *So how do I know which number is next? Is there a pattern? [Pause] Let's stop a minute. We had some people who were thinking across the chart instead.*

Teacher: *Go ahead, Ashley, talk that out.*

Ashley: *The numbers on top are going in columns.*

Teacher: *Can you give me an example?*

Ashley: *2, 4, 6, 8, like that.*

Teacher: *What do you think, Lashanette?*

Lashanette: *The first number is an odd number, the second one is even, the third one is odd …*

Teacher: *So what's the next number?*

Lashanette: *I don't know.*

An important part of this dialogue was that the teacher needed to be patient and encouraging. The students were encouraged to talk out their thoughts. Some students built on what other students were saying, even if they were slightly off track. The students got themselves back on track by continuing to address the problem.

Teacher: *Octavious.*

Octavious: *It goes like 1, 2, 3. Then goes 2, 4 and skip one to 8. Then it goes 3 and 6 and skip. Skip 2. So it's 4, 8, and skip 3, and then you go to the next number.*

Teacher: *What would it be?*

Octavious: *20.*

Teacher: *So we have 20 as a possibility, and I'm just going to start writing possibilities. We don't really know yet, but Octavious has a theory and he decided it would be a 20.*

Kathy: *You know how it goes 3 + 5 is 8 and then 8 + 7 is 15. So you would have to add 15 + 5 and make it 20, and then 20 + 7 would be 27.*

Teacher: *I'm still not sure how you got to 20.*

Kathy: *The pattern is that first it goes 5 and then 7, and so then I pick 5 to add. Next time, I'll add 7 and just keep switching back and forth.*

Teacher: *We have another theory. Thank you, Kathy. Reginica?*

Reginica: *I think it's 24.*

After some conversation with Reginica about her thinking, 24 is added to the list of possibilities.

Teacher: *Reginica is admitting that it is an educated guess, and that's okay.*

Octavious: *It's a skipping pattern.*

Teacher: *Now we have someone who is looking across [points to the chart on the whiteboard and indicates the idea of going across each row].*

The teacher waited as students talked with one another and thought of different possibilities. A lot of noise ensued, and then Consuelo started talking and the teacher said, "Say that again, Consuelo."

Consuelo: *It's 1 + 1 = 2 and 2 + 1 is 3. There's two people and one extra move they make, and that's three.*

Teacher: *Can you do that to the next row? Whatever you do, you have to do the same each time. You're getting closer.*

LaShanette: *The first one, they skip; and then they skip six; and the next one, they skip nine; and so I think it's 20.*

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Groups

On the fourth day of the lesson, we were working in the classroom across the corridor from the computer lab. The students sat in groups of four. Using their math journals and notes, the groups of students were given the task of organizing an oral presentation to explain the chart.

Using the chart, the groups looked for patterns. They started to realize that if the number of pairs were multiplied by itself and the number of people were added, the result would be the number in the minimum-number-of-moves column. Various students in groups gave reports explaining the arithmetic:

1 X 1 = 1. Take 1 and add 2 times 1 to it, and you get 3.

2 X 2 = 4. Take 4 and add 2 times 2 to it, and you get 8.

3 X 3 = 9. Take 9 and add 2 times 3 to it, and you get 15.

So they projected this pattern:

4 X 4 = 16. Take 16 and add 2 times 4 to it, and you get 24.

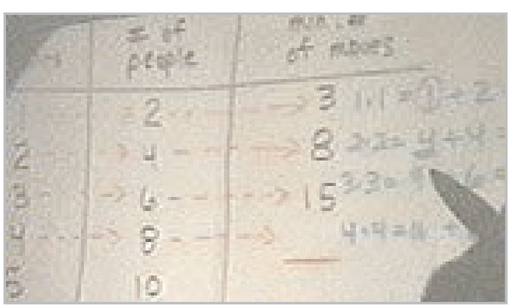
5 X 5 = 25. Take 25 and add 2 times 5 to it, and you get 35.

6 X 6 = 36. Take 36 and add 2 times 6 to it, and you get 48.

7 X 7 = 49. Take 49 and add 2 times 7 to it, and you get 63.

8 X 8 = 64. Take 64 and add 2 times 8 to it, and you get 80.

All the groups had completely filled in their chart with numbers, and now it was time to generalize.



Teacher: *Let's call any number in this first column n. Does that bother anyone?*

The students agreed that that was okay and added an n down at the bottom of the first column.

Teacher: *If the numbers in the first column are called n, what are the numbers in the second column called?*

One student: *p.*

Teacher: *But I only want to use n again.*

Another student: *Capital N.*

Teacher: *No, I don't want to do that either.*

A different student: *Double n.*

Teacher: *How do I write a double n?*

Another student: *m.*

A different student: *n-two.*

Another student: *n dot n.*

Teacher: *What does n dot n mean?*

Another student: *n squared.*

Teacher: *Didn't Kathy say "double n"? What's double n?*

Sam: *Two n's.*

Teacher: *How do I write two n's?*

Various students gave answers, and the teacher responded by saying, "No. . . no."

Student: *n + n.*

Teacher: *When I have double of things, which is n + n, I can also write it as 2n. It's not squared.*

At this point, "n" was at the bottom of the first column of the chart and "2n" was at the bottom of the second column of the chart.

Teacher: *Now, how do we write the stuff in the last column using n's?*

Student: *n X n and then add two n.*

Teacher: *If n equals the number of pairs and 2n is the number of people, then the minimum number of moves is n times n (or n squared) plus 2n.*

As the teacher wrote this statement, students could be heard repeating it quietly. They also wrote it in their notes.

Teacher: *Let's test our algebraic expression. Let's test it for three pairs. What is the minimum number of moves? … What do I do? Bryant?*

Bryant: *Three times three plus two times three. So you have nine plus six, and that equals fifteen.*

Kathy: *So we got it right?*

Teacher: *Yes, you got it right!*

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Assessment

Throughout the five days spent on the lesson, informal assessment took place by teacher observation. The students' math journals were another form of assessment.

In addition to the informal assessment and observation, the students were given the following written assessment:

State the algebraic expression generated in class that represents the minimum number of moves needed for n number of pairs of people to move from one side to the other.

If n = 11, what will be the minimum number of moves be? Show your arithmetic, and explain your answer in complete sentences.

Now that you have worked the Traffic Jam Activity using small plastic manipulatives and the computer Java applet and have looked at it algebraically, explain how the symbolic representation relates back to the six people moving on the stepping stones.

Conclusion

The culminating event of this lesson was the videotaping of groups of students performing the Traffic Jam activity with their bodies. The students were randomly put into groups of seven. Three students held green pieces of paper while another three held blue pieces of paper. The seventh student was the narrator for the performance. The groups had 20 minutes to practice. All groups successfully performed the problem for the video camera.

The last group to be videotaped concluded their performance with *"And that's the end!"*

Students who are beginning to develop algebraic thinking can benefit from experiences that are concrete. If given the time and the encouragement, students can formalize the mathematics behind an activity. Time and patience are the essential elements.

The Java applet enhances the lesson in two ways: (1) it extends the activity by engaging the students and (2) more important, it gives the students feedback. Students know if they have succeeded in finding the minimum number of moves. The applet keeps them working with the problem at a time when, had they been working with only their bodies or the plastic people, they would have given up long before.

The **combination** of all the parts of the lesson makes it a rich algebraic lesson. The students need (the manipulative experience) + (the technology experience) + (classroom discussion where they are free to work through their thoughts) + (the formalization of the mathematics). All these important parts of the lesson leave a lasting impression on students, helping them build the background needed for solid algebraic thinking.

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about the author

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references

1996 Math Forum Summer Institute

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student activities

You have read and followed the directions to the activity called Traffic Jam. Here's the problem:

There are seven stepping stones and six people. On the three left-hand stones, facing the center, stand three of the people. The other three people stand on the three right hand stones, also facing the center. The center stone is not occupied.

The challenge: exchanging places

Everyone must move so that the people originally standing on the right-hand stepping stones are on the left-hand stones and so that those originally standing on the left-hand stepping stones are on the right-hand stones, with the center stone again unoccupied.

The rules:

1. After each move, each person must be standing on a stepping stone.
2. If you start on the left, you may move only to the right. If you start on the right, you may move only to the left.
3. You may "jump" another person if the stone on the other side is empty. You may not "jump" more than one person.
4. Only one person may move at a time.

You have tried this activity using two manipulatives, your bodies and the small plastic people. Next try it using the computer. Go to this applet:

TrafficJam–Java Applet

Be sure to manipulate the various options that Mike Morton has made available, including these:

1. Level of difficulty
2 Show history

Look for a **pattern.** What do you see? Are there any **rules** to completing this activity successfully? What are they?

1. What if only two people and three spaces are involved?

   How many moves are needed for the two people to exchange positions?

2. What if four people and five spaces are involved?

   How many moves are needed for four people to exchange positions?

3. What about six?
4. What about eight?
5. What about 10?
6. Can you find a pattern for any number of people?

Another version of Traffic Jam can be found at the Math Forum.

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lesson plan

This activity is aligned with the NCTM Standards for grades 6–8: Algebra, Problem Solving, Reasoning and Proof, and Communication.

I first encountered this activity during the Math Forum's 1996 Summer Institute, where it was called Traffic Jam.

Here's the problem:

There are seven stepping stones and six people. On the three left-hand-stones, facing the center, stand three of the people. The other three people stand on the three right-hand stones, also facing the center. The center stone is not occupied.

The challenge: exchanging places

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3. You may "jump" another person if the stone on the other side is empty. You may not "jump" more than one person.
4. Only one person may move at a time.

Introducing the activity

Large-movement experience:

Each group of six students is given seven sheets of paper to use as stepping stones. Areas of the room are assigned to each group, and the activity begins.

Allow enough time for groups to try to find the minimum number of moves necessary to complete the task.

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Using Manipulatives

Simulating the activity:

Once the activity has been experienced as large movement, students use what they have learned to try it on a smaller scale.

Each group is given six small plastic figures or other objects with which to simulate the activity. As groups try to find the fewest number of moves necessary to complete the exchange of places, the teacher circulates among them to monitor the activity.

Using Technology

Interactive Web activity:

To simulate the Traffic Jam activity, Mike Morton wrote a Java applet for the Math Forum. Students can work individually, in pairs, in groups, or with one classroom display to investigate the problem further.

Traffic Jam Java Applet

Be sure to manipulate the various options that Mike has made available, including these:

1. Level of difficulty
2. Show history

After the students have tried the easy, medium, and hard levels, encourage them to look for a pattern.

Revisiting the Activity

At this point, students have investigated the problem using large manipulatives (their bodies), small manipulatives (plastic figures), and technology (the Java applet). Some students will discover the minimum number of moves for six people because they successfully complete the activity using the Java applet and the computer tells them that their answer is correct! Other students will not master the activity but may have gained a better understanding of the task.

Once more, have the students assemble in their groups of six, and repeat the activity using their bodies and the paper stepping stones. As they repeat the activity, observe groups that are successful and ask them to think of some "rules" to account for their success.

Extending the activity–looking

lesson plan continued

for patterns

Have the students sit down and think in terms of a pattern:

1. What if only two people and three spaces are involved?

   How many moves are needed for the two people to exchange positions?

2. What if four people and five spaces are involved?

   How many moves are needed for four people to exchange positions?

3. What about six people?
4. What about eight people?
5. What about 10 people?
6. Can you find a pattern for any number of people?

Writing the answer algebraically

Students can first make a data table using the information gathered so far. Columns might be made for just the number of pairs, the number of people, and the first three entries for the minimum number of moves.

Number of Pairs	Number of People	Minimum Number of Moves
1	2	3
2	4	8
3	6	15
4	4	...
5	10	...
6	12	...

Ask students: What patterns do you see? Are any relationships evident among the numbers in any of the three columns? Consider just the first and third column. What if we let n equal the number of pairs? Can we generate any of the numbers in the minimum-number-of-moves column?

Does 12 + 2(1) = 3?

Does 22 + 2(2) = 8?

Does 32 + 2(3) = 15?

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lesson plan continued

The completed table might look like this:

Number of Pairs	Number of People	Minimum Number of Moves	Another View
1	2	3	$1^2+2(1)=3$
2	4	8	$2^2+2(2)=8$
3	6	15	$3^2+2(3)=15$
4	8	24	$4^2+2(4)=24$
5	10	35	$5^2+2(5)=35$
6	12	48	$6^2+2(6)=48$
...	...	...	...
n	$2n$	$n^2+2(n)$	$n^2+2(n)=n(n+2)$

Extensions/ Resources

Solutions and Reflections, from participants in Math Forum Institute

Purpose of Algebra, from the Dr. Math Archives

Traffic Jams, a different problem by John Conway

Games That Interest John Conway

AlgebraCFun with Calendars, by Cynthia Lanius

The Million $ Mission, by Cynthia Lanius

“Middle School Mathematics Curriculum,” by Suzanne Alejandre

“Developing a Mathematics Curriculum That Integrates Activities, Exercises, Manipulatives and Technology:” a position paper for the NCTM’s Standards 2000 and Technology Conference held in Washington, D.C., June 5B6, 1998.

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